THE ANALYST;

OR,

MATHEMATICAL MUSEUM.

VOLUME I. NUMBER IV.

ARTICLE XI.

SOLUTIONS TO THE QUESTIONS PROPOSED IN ARTICLE X.

SOLUTION TO QUESTION I.

By Richard Pearse and Henry White, Baltimore.

THERE being an hundred towns, and no three of them in the same straight line; it follows that 99 roads will be necessary for any particular Mathematician to visit all the others; and therefore $100\times99=9900=$ the whole number of visits: but each road will then be twice travelled over, and therefore $\frac{9900}{2}$ or 4950 will be the number of roads sought.

Another solution by Thomas Kimber, Chester county, Pennsylvania.

It is evident that the first Mathematician A, must have a road to each of the others, that is, he must have 99 roads; the second B, using the road made from A to him, needs but 98 new roads; the third C but 97, &c. to the last, who having received visits from each of the others, will consequently require no new road. Therefore the whole number of roads required will be the sum of a decreasing series in arithmetical progression, of which the first term is 99, common difference 1, and number of terms 99, which sum is evidently $99\times\frac{99+1}{2}=4950$.

N. B. Most of the contributors have used the method of solution exhibited by Thomas Kimber. Ed.

A different solution, by John Gummere, and Seth Smith, New Jersey.

Since there must be one road between every two mathematicians, it is evident that the number of the combinations of two things in 100 will be the number required $=100\times\frac{100-1}{2}=4950$.

QUESTION II.

Solved by F. R. Hassler, Professor of Mathematics at the Military College, West Point, State of New York; and by Joshua Gilbert, of Morrisville, Falls of Delaware.

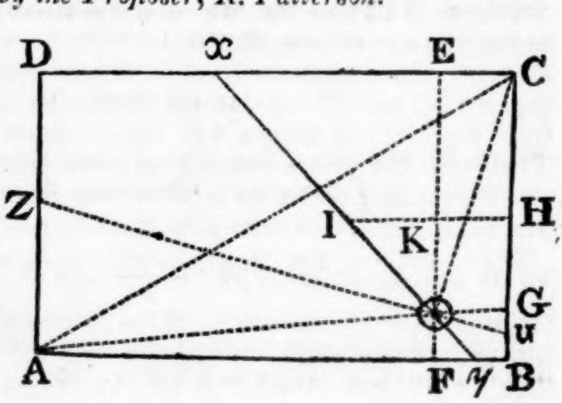
Let n represent the number of divisions made in the circumference of the circle, then $\frac{n}{2}$ will be the divisions in a semicircle, and therefore $\frac{n}{2}-1=$ the number of angles that can be formed on one side of any fixed point in the circumference, each of these angles being less than two right angles; and there being n such fixed points in all, there will be $\left(\frac{n}{2}-1\right)n$ angles at the centre, each less than two right angles; and as $n=360$, therefore $\left(\frac{n}{2}-1\right)n=(180-1)\times360=64440$ as required.

A different solution, by Daniel Smith, Burlington, New Jersey.

It is plain from the data of the question, that there will be 360 angles of 1°, there will also be 360 angles of 2° each, of 3°, 4° &c. to 179° which, by the question, is the greatest angle to be taken into consideration; hence the number required is $360\times179=64440$.

QUESTION III.

Solved by the Proposer, R. Patterson.



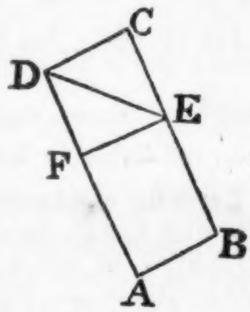
Let ABCD represent the oblong piece of land, and O the spring of water within the same, 250 perches from the corner A, and 170 from the opposite corner C; bisect BC in H, and draw HI perpendicular to BC and $=\frac{1}{3}$ of AB $=100$; then x IOy will be the par-

tition-line required. Draw FOE parallel to BC, and OG parallel to AB.

1. In the right-angled triangle ABC, are given the two legs, to find the hypothenuse and oblique angles. 2. In the oblique triangle AOC there will be given the three sides, to find the angle ACO. 3. In the right-angled triangle COG there will be given the hypothenuse and oblique angles, to find the legs. 4. By similar triangles, OK : KI : : OE : Ex, and OK : KI : : OF : Fy. According to the above calculations, Cx=176.46, By=23.56, and xy=251.7. By inspection of the figure it will appear that another line, zu, may be drawn through O so as to divide the plot into the parts required; but this line will be much longer than xy, which is therefore the partition-line required.

QUESTION IV.

Solved by F. R. Hassler, West Point.



Let ABCD be a vertical section of the cylinder through its axis, and intersecting the upper surface of the fluid in the straight line DE; and draw EF parallel to CD. But a=the measure of the empty part CDE=10, and $2a$=the content of the cylinder CF; also put τ=the tangent of the angle CED which is the inclination of the cylinder to the horizon, or the complement of its inclination to the vertical, and let x be the content of AC.

Now making CE radius, CD is the tangent of CED, and BC is double this tangent by the question, that is, CE is to CB, as 1 to 2τ, which ratio is evidently the same as that of CF to CA, therefore as $1 : 2\tau : : 2a : 4a\tau = x =$the content required; if CED$=40°$, we have $x=40\tau=33.96398$; but if CED $=50°$, then $x=40\tau=47.67015$.

QUESTION V.

Solved by William Child, Pottsgrove, Pennsylvania, and Josiah Tatum, near Woodbury, New Jersey.

ft ft

Put a=400= the length, and b=160= the breadth of the

A ft

original survey, c=1000×160=160000= the additional area, and let x=the distance between the old and new boundaries, then $2x+a$, and $2x+b$ will be the length and breadth of the

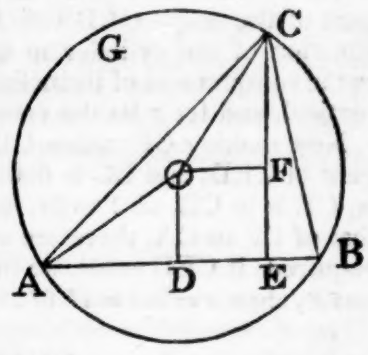
whole; otherwise, its area will be expressed by $(2x+a)\times(2x+b)$, or by $4x^2+(2a+2b)x+ab$; from which deducting $ab=$ the content of the original survey there remains $4x^2+(2a+2b)x=c$; in numbers, $x^2+280x=40000$, from which we obtain $x=\sqrt{59600}-140=104.13$ perches, the distance required.

General Rule for such Problems, by Peter Spangler, York Town, Pennsylvania.

Add together the length and breadth of the old settled tract, and take one fourth of the sum; to the square of which fourth part add one fourth part of the preemption-right in perches; and from the square root of this last sum take one fourth of the sum of the given length and breadth, and the remainder will be the breadth of the new survey; thus, $\overset{a}{1000}=\overset{p}{160000}, \frac{160+400}{4},=140,$ and $140\times140=19600$; again, $\frac{160000}{4}+19600=59600$, and $\sqrt{59600}$, $=244.13$; lastly, $244.13-140=104.13=$ the breadth required.

QUESTION VI.

Solved by Seth Smith, Burlington, New Jersey.



Let ABC represent the triangle, with its circumscribing circle ABCG, of which the radius AO or OC is 20; suppose OD, CE, perpendicular to AB, and OF parallel to it.

Because the angle ACB is given $=61^{\circ}.30'$, its equal AOD is also given, from which, with AO, we find AD,DO; and the double of AD is AB. Again, the given area divided by AD gives CE, from which deducting EF=OD, we have CF; and from OF and CF we obtain OF, or its equal DE; hence AE,EB become known; and with these, and the perpendicular CE, we obtain the sides AC, and BC: in this manner we find AB=36.15, BC=26.58, AC=38.95.

N. B. Thomas Whitaker, York Town, Pennsylvania, solved this question elegantly by Problem V. Appendix to Simpson's Algebra. Ed.

QUESTION VII.

Solved by Ebenezer R. White, Danbury, Connecticut, and Hezekiah Boone, Northumberland County, Pennsylvania.

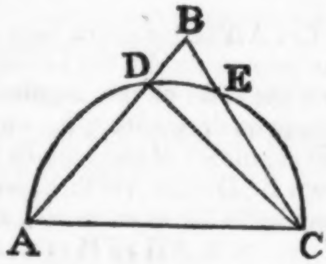
Let h=the given hypothenuse, b and p the base and perpendicular; then by the question $bp=(b-p)^2=b^2-2bp+p^2$, whence $3bp=b^2+p^2=h^2$, consequently, $bp=\frac{h^2}{3}$, and $\frac{1}{2}bp=\frac{h^2}{6}$ =the area required.

N. B. John D. Craig, Baltimore, solved this question by an easy geometrical construction. Ed.

QUESTION VIII.

Solved by the Proposer, William Lenhart.

Let ABC be the triangle required. On the given base AC describe the semicircle ADEC, the circumference of which cuts the sides AB, BC, in the points D and E. Let AC $=a$, AD : DB : : $c:d$, CE : EB : : $m:n$, AD$=cx$, and CE$=mz$, then will BD$=dx$, and BE$=nz$; also AB $=(c+d)x$, BC$=(m+n)z$, AB×BD $=(c+d)\times dx^2$, and BC×BE$=(m+n)nz^2$.



But by Cor. Prop. 36th B. III. Simpson's Euclid, AB. BD=CB.BE, whence $(c+d)dx^2=(m+n)nz^2$. Let now CD be drawn, which by Prop. 31. B. III. Euc. Elem. will be perpendicular to AB, and consequently $\overline{AC}^2-\overline{BC}^2=\overline{AD}^2-\overline{DB}^2$, that is, $a^2-\overline{m+n}^2.z^2=(c^2-d^2)x^2$, whence $x^2=\frac{a^2-\overline{m+n}^2.z^2}{c^2-d^2}=\frac{\overline{m+n}.nz^2}{\overline{c+d}.d}$,

and of consequence $z=\frac{a\sqrt{d}}{\sqrt{\overline{m+n}^2.d+\overline{c-d}.\overline{m+n}.n}}$, from which every thing may be found.

N. B. Mr. Whitaker's solution was a little different, and very neat; Mr. Hassler's algebraic solution was also curious. Ed.

Construction of the same by John Gummere.

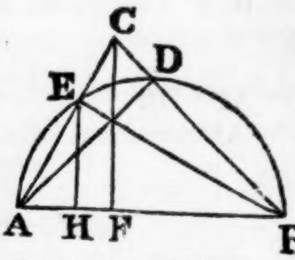
Let AB be the given base of the triangle, and produce it each way to D and E, so that AB may be to AD and BE in the given ratios of the segments of the sides, and on AB, BD, describe the semicircles ACE, BCD, and let their circumferences intersect in C: join AC, BC, and ABC will be the triangle required.

For join AI, CD, and AIB, BCD, being each an angle in a semicircle are right angles, consequently the triangles AIB, BCD are similar, and BI : IC : : BA: AD; in like manner it is proved that AH is to HC as AB is to BE.

Calculation. From F and G the centres of the semicircles BCD, ACE, draw FC, GC, and with the given base AB and ratios of the sides find AD and BE, from which will be known all the sides of the triangle FCG, to find the angle F; and in the triangle AFC are given AF, FC, and angle F to find the side AC: in a similar manner we find the side BC.

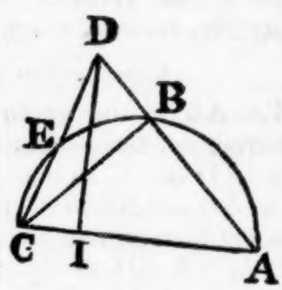
N. B. In the same elegant manner the problem was constructed by Samuel Cowgill, Daniel and Seth Smith, John Coope, and the proposer, William Lenhart. Ed.

Another construction of the same by John D. Craig, Baltimore.



Let AB be the given base on which the semicircle AEDB is described; m to n the ratio of the segments of the side to be drawn from A; s to r that of the segments of the side to be drawn from B. Divide AB in F, so that AF may be to BF as sn to mr; and AF in H, so that AH : FH : : $n : m$. Erect the perpendiculars HE and FC, and through A,E, draw AE meeting FC in C; also draw BDC, then is ABC the triangle required: for if AD and BE be joined, the angles at D,E and F being right angles, the perpendiculars CF, AD, and BE, must pass through the same point, therefore AE× CD : BD×CE : : AF : BF (23. iv. Simpson's Elements Geom.) Moreover, since AE : EC : : AH : HF : : $n : m$, the construction is manifest.

Another Algebraic solution of the same by Nathaniel Bowditch, of Salem, Massachusetts.



Let ADC be the triangle whose base AC$=a$, AB$=y$, AD$=by$, CE$=x$, CD $=cx$, and let DI be perpendicular to CA. Then by the similar triangles ABC, AID we have CA : AB : : AD : AI$=\frac{byy}{a}$, and in a similar manner, CI$=\frac{cxx}{a}$ ∴ CA$=\frac{byy+cxx}{a}=a$. Again, sin. DB×DA=DE×DC, we have $b.\overline{b-1}$.

$yy=c.\overline{c-1}.xx$. The value of y^2 found from this and substituted in the former gives $cx=a\sqrt{\frac{c.(b-1)}{b+c-2}}$; whence $by=a\sqrt{\frac{b.(c-1)}{b+c-2}}$, which are the sides required.

QUESTION IX.

Solved by John Capp, Harrisburg.

Put the base of the triangle$=10=a$, the perpendicular $=x$, and the hypothenuse $=vx$, and we shall have $v^2x^2-x^2=a^2$; and, per question, $vx\times x=vx^2=200=2a^2$, therefore $2v^2x^2-2x^2=2a^2=vx^2$; from which $2v^2-2=v$, or $v2-\frac{1}{2}v=1$; and completing the square, $v=\frac{1}{4}\times(\sqrt{17}+1)=12807764$. And because $v^2x^2=a^2+x^2=2va$, therefore $x^2=2va^2-a^2=a^2\times(2v-1)$; whence $x=a\times\sqrt{2v-1}=12.496234$, and hence $vx=16.004914$.

N. B. Mr. M'Ginnis' solution was nearly similar, and equally ingenious. Ed.

A different solution by William Douglas, near Trenton, N. Jersey.

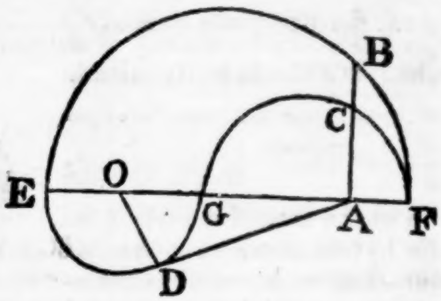
Put $a=$ the given base $=10$, $b=$ the given product$=200$, and $x=$the perpendicular; then $a^2+x^2=\frac{b^2}{x^2}$, that is, $x^4+a^2x^2=b^2$, from which $x=\sqrt{(b^2+\frac{1}{4}a^4)^{\frac{1}{2}}-\frac{1}{2}a^2}=\sqrt{206.1552-50}=12.496=$ the perpendicular, and $200\div12.496=16.005=$the hypothenuse.

N. B. According to this second method the problem was solved by William Cherington, Reading, and Jacob Conklin, Bergen County, New Jersey.

Another solution of the same, by Thomas Whitaker, York Town, Pennsylvania.

Let the given rectangle $=a$, the given base $=b$: and let $x=$ the hypothenuse: then $\sqrt{x^2-b^2}=$ the perpendicular. But $x\times\sqrt{x^2-b^2}=a$ by the question; whence $x=\sqrt{(a^2+\frac{1}{4}b^4)^{\frac{1}{2}}+\frac{1}{2}b^2}=16$ nearly, the hypothenuse. And $\sqrt{x^2-b^2}=12.49=$the perpendicular.

The same Problem constructed by the Proposer, Charles Richards.



With the radius OD, equal to the given base, describe the semicircle EDG, and make the rectangle contained by the half of OD and the perpendicular DA equal to the given rectangle; join AO, and having produced it both ways to E and F, make AF equal to the half of OD; on EF, GF, describe the semicircles EBF, GCF, and making ACB perpendicular to EF, AB and AC will be the hypothenuse and perpendicular of the triangle required.

For the difference of the squares of AB and AC is evidently equal to the difference of the rectangles FA.AE and EA.AG, that is, to the rectangle FA.EG, that is, to the square of OD. Again, AE is to AD as AD to AG, therefore AE.AF or the square of AB is to AD.AF, as AD.AF or the given rectangle is to AG.AF or the square of AC; and therefore BA.AC is equal to the given rectangle. Q.E.D.

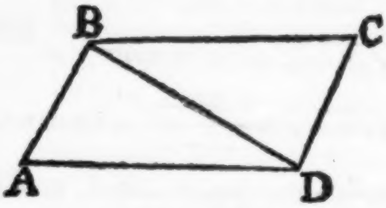
QUESTION X.

Or, the Prize Question, solved by Nathaniel Bowditch.

Put B= the supplement of the angle ABC, and $60^\circ - B$ = the supplement of ADC, $a=6$, $b=12$, $c=8$, $d=10$, cos. $B=x$, cos. $(60^\circ - B) = \frac{1}{2}x + \sqrt{\frac{3}{4}\cdot\overline{1-xx}}$, thus the square of the diagonal AC $= aa+bb+2abx = cc+dd+2cd(\frac{1}{2}x+\sqrt{\frac{3}{4}\cdot\overline{1-xx}})$, which by putting $\frac{aa+bb-cc-dd}{cd} = \frac{1}{5} = e$, and $\frac{2ab}{cd} - 1 = \frac{4}{5} = f$, becomes $e+fx = \sqrt{3\cdot\overline{1-xx}}$, hence $x = -\frac{ef+\sqrt{9+3\cdot\overline{ff-ee}}}{ff+3} = 0.8588832$ = cos. $30^\circ\ 48'\ 31''$ $\because$ $ABC = 149^\circ\ 11'\ 29''$ and $ADC = 150^\circ\ 48'\ 31''$. Whence the diagonal $AC = 17.42639$.

N. B. In the same manner, very nearly, the question was solved by Seth Smith, and Samuel Cowgill, Burlington, New Jersey.

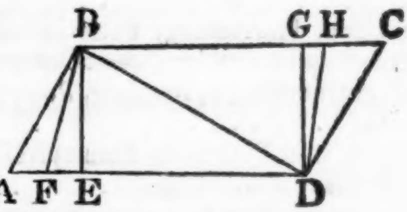
Another solution of the same, by Daniel Smith, jun. Burlington, N. J.



Put $n=\sqrt{\frac{3}{4}}=$ sin. 60°. (=A+C) its cosine$=\frac{1}{2}$, $x=$ cos. BCD, then its sine$=\sqrt{1-x^2}$, and the cos. BAD$=\frac{x}{2}+n\sqrt{1-x^2}$. In the triangle BCD, we have (by Sim. Euc. Pl. Trig. Prop. 5.) As $2.BC.CD : BC^2+CD^2-BD^2 :: 1 :$ cos. BCD; that is as $192 : 208-BD^2 :: 1 : x$, whence $BD^2=208-192x$; also in the triangle BAD, we have (by the same prop.) as $120:136-BD^2::1:\frac{x}{2}+n\sqrt{1-x^2}$, therefore $136-60x-120n\sqrt{1-x^2}=BD^2=208-192x$, or $132x-72=123n\sqrt{1-x^2}$, which divided by 12 gives $11x-6=10n\sqrt{1-x^2}$, then squaring, and writing $\frac{3}{4}$ for n^2 we get $121x^2-132x+36=75-75x^2$, or $x^2-\frac{132}{196}x=\frac{39}{196}$, whence by completing the square and extracting the root, $x=.895635$ the natural cosine of 26° 24′ 35″$=\angle$BCD, whence the area$=37.947$.

N. B. Mr. Gummere's solution was not very different, and equally ingenious: he put $x=$ cos. A, and found the quadratic $x^2+\frac{6x}{49}=\frac{39}{49}$ or $x=\frac{8\sqrt{30}-3}{49}=$ cos. 33° 35′ 24½″, and the area $=37.9473$.

Solution of the same Prize Question, by Joseph Clay, Philadelphia.



The question in a general form is, Given the four sides of a trapezium, and the sum of two opposite angles, to find the area.

Put $AB=a=6$, $AD=b=10$, $BC=c=12$, $CD=d=8$; join BD, draw BE,DG perpendicular to AD,BC, and make the angles BFD, BHD each equal to the sum of the two smaller angles A,C, (in the present case=60°).

Put $BF=y$, $DH=z$, $AF=p$, $CH=q$, Sine BFF to rad. $1=s$, its cosine t; then $BE=sy$, $FE=ty$, $DG=sz$, $HG=tz$; also by sim. tri. $a:d::y:q::p:z$, whence $q=\frac{dy}{a}$, $z=\frac{dp}{a}$. Now $BD^2=a^2+b^2-2b\times\overline{p+ty}=c^2+d^2-2c.\overline{q+ty}$, therefore $c^2+d^2-a^2-b^2=$

$f^2=\dfrac{2cdy+2cdt-2abp-2abty}{a}$: but $p+ty=\sqrt{a^2-s^2y^2}$, and $y+tp=t\sqrt{a^2-s^2y^2}+s^2y$ therefore $f^2=\dfrac{2cds^2y+\overline{2cdt-2ab}\times\sqrt{a^2-s^2y^2}}{a}$, and $\sqrt{a^2-s^2y^2}=\dfrac{2cds^2y-af^2}{2ab-2cdt}$, from which, by a quadratic, y, and consequently the area $s\times\dfrac{by+cz}{2}$, may in all cases be found.

But in the present case the equation for y becomes $\sqrt{a^2-\frac{3}{4}y^2}=6y-18$, from which by squaring, &c. we have $y-\dfrac{144}{49}=\dfrac{\sqrt{1920}}{49}$: also $6z=8p=4\sqrt{144-3y^2}-47$, and therefore the area sought $=\dfrac{\sqrt{3}}{2}\times\overline{57+6z}=\dfrac{\sqrt{3}}{2}\times(y+4\sqrt{144-3y^2})=\dfrac{\sqrt{3}}{2}\times(49y-144)=\dfrac{\sqrt{3}}{2}\times\sqrt{1920}=12\sqrt{10}=37.94733$, &c.

ACKNOWLEDGMENTS, &c.

A list of the names and places of residence of the ingenious gentlemen who have favoured the editor with solutions to the questions proposed in No. III. Art. 10. The figures annexed to the names refer to the questions answered by each as numbered in the work.

Hezekiah Boone, Northumberland county, Pennsylvania, ...	1	2	3	4	5		7			
Nathaniel Bowditch, Salem, Massachusetts, ...	1	2	3	4	5	6	7	8	9	10
John Capp, Harrisburg, Pennsylvania,									9	10
Joseph Chaplin, Rowley, Massachusetts,					5					10
William Cherington, Reading, Pennsylvania, ...	1	2	3		5				9	
William Child, Pottstown, Pennsylvania,			3		5	6	7		9	
Joseph Clay, Philadelphia, ...										10
Jacob Conklin, Bergen county, N. Jersey,			3	4	5		7		9	10
John Coope, Philadelphia, ...	1	2	3	4	5	6	7	8	9	10
Samuel Cowgill, near Burlington, New Jersey, ...	1	2	3	4	5	6	7	8	9	10
John D. Craig, Baltimore, ...	1	2	3	4	5	6	7	8	9	10
Benjamin Davies, Reading, Pennsylvania, ...	1								9	10
William Douglas, near Trenton, New Jersey, ...	1	2	3		5	6	7	8	9	
Joshua Gilbert, Morrisville, Falls of Delaware, ...	1	2	3		5	6	7	8	9	

James M'Ginnis, Harrisburg, Pennsylvania,	1	2	3	4	5	6	7	8	9	10
John Gummere, near Burlington, New Jersey,	1	2	3	4	5	6	7	8	9	10
F. R. Hassler, West Point, New York State,	1	2	3	4	5	6	7	8	9	
Thomas Kimber, Chester county, Pennsylvania,	1				5	6	7		9	
William Lenhart, Baltimore,	1	2	3	4	5	6	7	8	9	
Robert Patterson, Philadelphia,			3	4						
Richard Pearse, Baltimore,	1		3	4	5	6	7		9	
Charles Richards, Reading, Pennsylvania,	1	2	3	4	5	6	7	8	9	10
Daniel Smith, Burlington, New Jersey,	1	2	3	4	5	6	7	8	9	10
Seth Smith, Burlington, New Jersey,	1	2	3	4	5	6	7	6	9	10
Peter Spangler, York Town, Pennsylvania,			3		5		7		9	
Josiah Tatum, near Woodbury, New Jersey,			3		5	6	7		9	
E. R. White, Danbury, Connecticut,			3				7		9	10
Henry White, Baltimore,	1		3	4	5	6	7		9	
Thomas Whitaker, York Town, Pennsylvania,	1	2	3	4	5	6	7	8	9	10

☞ The prize of six dollars has been awarded to Joseph Clay, Philadelphia.

No Geometrical construction of this prize question has been received by the editor; he would willingly publish his own method of constructing the problem geometrically, were he not desirous of exciting geometers to a research equally curious and elegant: the question is therefore re-proposed in the present Number, Article 15.

ARTICLE XII.

CONTAINING SOLUTIONS TO TWO QUESTIONS PROPOSED IN ARTICLE X. VIZ. THE 1st. AND 3d.

Besides the solutions given to the 1st question in the preceding number, it is susceptible of several others of different kinds, particularly the two following:

Solution to Question I. 21, *Article X.*

BY THE PROPOSER, ROBERT ADRAIN.

Let N be any number of mathematicians, and R the corresponding number of roads. Suppose N to be augmented by its

finite difference, $\Delta N=1$, and the number of new roads on this account will evidently be N or its equal $\Delta N.N$, for the new mathematician must have a road to each of the former N; therefore $\Delta R=\Delta N.N$, which equation integrated in the usual way gives $R=C+\frac{1}{2}N-1.N=C+\frac{1}{2}(N-1).N$: but when $N=0$, or $N=1$, we know that $R=0$, in which case the general equation $R=C+\frac{1}{2}(N-1).N$ becomes $0=C$, therefore in general $R=\frac{1}{2}(N-1).N$, and in the case proposed $R=\frac{1}{2}(100-1).100=4950$.

A different solution by the same.

Let x be any number of mathematicians, and ψx the corresponding number of roads. Suppose x to be augmented by unity, and by similar functions, the corresponding number of roads will be $\psi(x+1)$; but the number of new roads being evidently x, we have therefore $x+\psi x=\psi(x+1)$. This equation in fluxions is $\dot{x}+\dot{x}\psi'x=\dot{x}\psi'(x+1)$ therefore $1+\psi'x=\psi'(x+1)$, of which the fluxion is $\dot{x}\psi''x=\dot{x}\psi''(x+1)$, or $\psi''x=\psi''(x+1)$. Now two similar functions of two unequal quantities x and $(x+1)$ cannot be universally equal while those unequal quantities vary, unless such similar functions be independent of the variations; that is, unless such similar functions be constant quantities, therefore $\psi''x=c$, whence $\dot{x}\psi''x=c\dot{x}$, and integrating, $\psi'x=b+cx$; again $\dot{x}\psi'x=b\dot{x}+cx\dot{x}$, and the fluents are $\psi x=a+bx+\frac{1}{2}cx^2$; but since when $x=0$, $\psi x=0$, therefore $\psi x=bx+\frac{1}{2}cx^2$.

When $x=1$, or 2, we know that $\psi x=0$ or 1, whence we have the two equations $0=b+\frac{1}{2}c$, and $1=2b+2c$, from which $b=-\frac{1}{2}$, $c=1$, and therefore the number of roads required $=\psi x=-\frac{1}{2}x+\frac{1}{2}x^2$, or $\psi x=\dfrac{x.\overline{x-1}}{2}$ as before.

In a similar manner we may find the value of ψx in the equations of similar functions $x^2+\psi x=\psi(x+1)$, $x^3+\psi x=\psi(x+1)$, &c. which is equivalent to finding the sums of the series of integer squares, cubes, &c. of which the roots are in arithmetical progression; and in general we may resolve in the same manner the equation $\phi x+\psi x=\psi(x+b)$, in which $\phi x=a+bx+cx^2+dx^3+ex^4$ &c. to n terms.

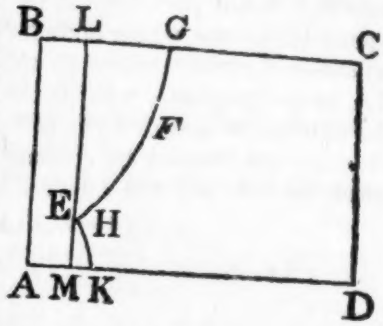
Solution to Question III. 23, Article X.

BY ROBERT ADRAIN.

Although the author of this question intended that the boundary should be a straight line, yet, as there is nothing in the question itself on which this supposition is founded, we shall consider the question without regard to such a limitation, and endeavour to discover the *nature* and *position* of the shortest line

possible passing through the well and cutting off one third of the area.

Let ABCD be the rectangular field, E the place of the well, and GFEHK the line or boundary required, and which must therefore fulfil the following conditions. I. The line or boundary sought GEK must pass through the given point E, and be terminated by the perimeter ABCDA of the field. II. It must cut off an area GEKABG equal to one third of the field ABCD. III. This boundary GEK must be the least possible that can fulfil the two preceding conditions.



Draw LEM parallel to AB one of the less sides of the field, and because the rectangle AL is by computation less than one third of the field, it is manifest that some part of the boundary GEK must fall without the rectangle AL: but since EL, EM are the least lines that can pass from E to the sides BC, AD, it is plain that neither branch EFG, EHK, of the boundary can fall within the rectangle AL; for in such case the inclosed area AG might be augmented by diminishing the boundary, that is by making the part of the boundary within AL to approach to a coincidence with EL or EM; one of the two branches EFG,EHK, must therefore fall without AL, and the other branch must coincide with EL or EM, or also fall without AL. Farther, the branch EFG or EHK that falls without AL cannot be convex towards LM, otherwise a greater area AG might be inclosed by bringing such branch to a coincidence with its chord. There is no difficulty in perceiving also that both the branches EFG, EHK must fall without AL; for if EM were supposed a part of the boundary, and that this part changed its position by the indefinitely small augmentation of AM, the variations of AM, of the area EHKM, and consequently of the area EFGL, and of the curve EF would be of the first order, but the variation of EM (as being perpendicular to AM) would be of the second order, and therefore its augmentation would be indefinitely less than the corresponding diminution of EFG; hence it appears that the boundary sought consists of two arcs or branches AEG, EHK both concave towards LM. All this reasoning however refers to the case in which the boundary sought is supposed to meet the greater side AD, BC; the nature and position of the boundary may be determined in a similar manner when the boundary is supposed on any one or two of the sides of the field AC.

We are at liberty to consider the nature of each branch of the boundary separately; for if the branch EFG were not the least line that could pass from E to BGC and inclose the determinate space ELG, the whole boundary could not be a minimum. The question therefore comes to this; supposing the area ELG, and EL to be invariable, what is the nature and position of the curve EFG when its length EFG is a minimum? This is a question that is easily determined by common geometry; in fact, it has been often solved, and the curve EFG is known to be a circular arc, cutting LG perpendicularly in G, its sine being the given line EL: from which it follows that the whole boundary sought consists of two circular arcs EFG, EHK, cutting the sides BC, AD perpendicularly in G and K; and of course these arcs have for their sines, EL, and EM, and their centres are in the sides BC, AD.

The sum of the areas of the two semisegments ELG, EMK, is given, and the sines of the arcs EG, EK are given, we have therefore to find two circular arcs, such that their sines may be given, that the sum of the areas inclosed by the arcs with their sines and versed sines may be given, and that the sum of those circular arcs may be a minimum.

The values of these arcs, and of their radii may be determined various ways; the similarity of circles furnishes an easy method, we shall therefore give a sketch of it.

Let a and b denote the given sines EL, and EM; s the given sum of the semisegments ELG, EMK; x and y the sines of two circular arcs z and u similar to EG, EK, and having their common radius unity. Then the areas of the two semisegments of which the arcs are z and u, sines x and y, and radius unity, are $\frac{1}{2}(z-x\sqrt{1-x^2})$ and $\frac{1}{2}(u-y\sqrt{1-y^2})$; and the areas of similar figures being as the squares of their homologous sides, the areas ELG, EHK will be found by saying as $x^2 : a^2 : : \frac{1}{2}(z-x\sqrt{1-x^2})$ $:\text{ELG}=\frac{a^2}{2x^2}(z-x\sqrt{1-x^2})$; in like manner $\text{EMK}=\frac{y^2}{2b^2}(u-y\sqrt{1-y^2})$ and therefore,

$$\text{(A.)} \qquad \frac{a^2}{2x^2}(z-x\sqrt{1-x^2})+\frac{b^2}{2y^2}(u-y\sqrt{1-y^2})=s.$$

Again, the homologous sides of similar figures being proportionals, as $x : z : : a : \frac{az}{x}=\text{EFG}$, and $\frac{bu}{y}=\text{EHK}$ and therefore

$$\text{(B)} \qquad \frac{az}{x}+\frac{bu}{y}=\text{ minimum.}$$

The equations (A) and (B) being put into fluxions, and $\frac{\dot{x}}{\sqrt{1-x^2}}, \frac{\dot{y}}{\sqrt{1-y^2}}$ being substituted for $\dot{z}$ and $\dot{u}$, we have

$$(A') \qquad \left\{\frac{a^2}{\sqrt{1-x^2}}-\frac{a^2}{x^3}(z-x\sqrt{1-x^2})\right\}\times\dot{x}=\left\{\frac{b^2}{\sqrt{1-y^2}}-\frac{b^2}{y^3}\right.$$
$$\left.(u-y\sqrt{1-y^2})\right\}\times-\dot{y},$$

$$(B') \qquad \left\{x\frac{a}{\sqrt{1-x^2}}-\frac{az}{x^2}\right\}\times\dot{x}=\left\{y\frac{b}{\sqrt{1-y^2}}-\frac{bu}{yz^2}\right\}\times-\dot{y},$$

Which two equations reduced to common denominators give us the two following:

$$(A'') \qquad \frac{a^2}{x^3\sqrt{1-x^2}}\times(x-z\sqrt{1-x^2})=\frac{b^2}{y^3\sqrt{1-y^2}}\times(y-u\sqrt{1-y^2}),$$

$$(B'') \qquad \frac{a}{x^2\sqrt{1-x^2}}\times(x-z\sqrt{1-x^2})=\frac{b}{y^2\sqrt{1-y^2}}\times(y-u\sqrt{1-y^2});$$

and the former of these divided by the latter produces

$$(C) \qquad \frac{a}{x}=\frac{b}{y}.$$

This last equation teaches us that the radii of the circular arcs EG and EK are equal, for $\frac{a}{x}$ and $\frac{b}{y}$ are evidently the values of those radii; from which we may remark that the curvature of the boundary GFEHK is uniform throughout.

It would now be a matter of no difficulty to ascertain the values of x and y, &c. from the equations (A) and (C) by means of any of the common methods of approximation.

It appears from the preceding investigation, that there is no continuous curve that will completely satisfy the conditions of the problem; for though the curvature of the branches EFG, EHK be the same, yet as they are referred to different centres, the continuity of the curvilineal boundary is evidently interrupted in E: we may however find continuous curves that will fulfil the conditions, within any required limits of accuracy, by describing a curve through any assumed number of fixed points in the discontinuous boundary, according to the common rules of the differential method.

On this head we shall only observe that a line of the nth order may be described through a number of points expressed by $\frac{1}{2}n.(n+3)$; as we have just investigated by the method of Similar Functions.

ARTICLE XIII.

Solution of Mr. Patterson's Prize Question for correcting a survey, proposed in No. II. page 42, No. III. page 68, by Nathaniel Bowditch, to whom the Editor has awarded the prize of ten dollars.

The principles, on which the solution of the Prize Question, page 42, ought to depend, appear to me to be these.

1. That the error ought to be apportioned among *all* the bearings and distances.

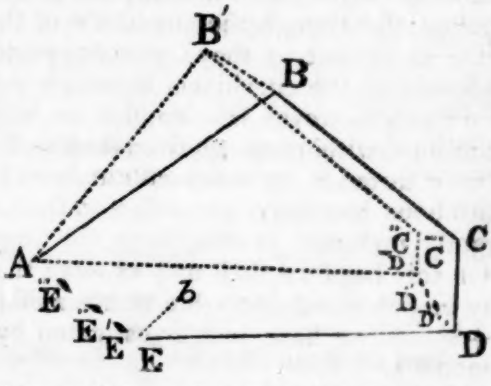
2. That in those lines in which an alteration of the measured distance would tend considerably to correct the error of the survey, a correction ought to be made; but when such alteration would not have that tendency, the length of the line ought to remain unaltered.

3. In the same manner, an alteration ought to be made in the observed bearings, if it would tend considerably to correct the error of the survey, otherwise not.

4. In cases where alterations in the bearings and distance will both tend to correct the error, it will be proper to alter them both, making greater or less alterations according to the greater or less efficacy in correcting the error of the survey.

5. The alterations made in the observed bearing and length of any one of the boundary lines ought to be such that the combined effect of such alterations may tend wholly to correct the error of the survey.

Suppose now that ABCDE represent the boundary lines of a field, as plotted from the observed bearings and lengths, and that the last point E, instead of falling on the first A, is distant from it by the length AE. The question will then be, what alterations BB′, CC″, DD‴, &c.



must be made in the positions of the points B, C, D, &c. so as to obtain the most probable boundaries AB′ C″ D‴ A?

If AB′ be supposed to be the most probable bearing and length of the first boundary line, the point B would be moved through the line BB′, and the following points C, D, E, would in conse-

quence thereof be moved in equal and parallel directions to C′, D′, E′, and the boundary would become AB′C′D′E′. Again, if by correcting in the most probable manner the error in the observed bearing and length of BC, (or B′ C′) the point C′ be moved to C″, the points D′ and E′ would be moved in equal and parallel directions to D″ and E″, and the boundary line would become AB′ C″ D″ E″. In a similar manner, if by correcting the probable error in the bearing and length of CD, (or C″ D″) the point D″ be moved to D‴, the point E″ would be moved in an equal and parallel direction to E‴, and the boundary would become AB′ C″ D‴ E‴. Lastly, by correcting the probable error in the bearing and length of the line DE, (or D‴ E‴) the true boundary AB′ C″ D‴ A would be obtained.

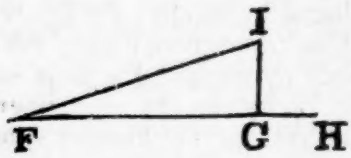
If we suppose the lines BB′, CC″, DD‴, &c. to be parallel to AE, it would satisfy the second, third, fourth, and fifth, of the preceding principles. For the change of position of the points B, C, &c. being in directions parallel to AE, the whole tendency of such change would be to move the point E directly towards A, conformably to the fifth principle, and by inspecting the figure it will appear that the second, third, and fourth principles would also be satisfied. For, in the first place, it appears that the bearing of the first line AB would be altered considerably, but the length but little. This is agreeable to those principles; because an increase of the distance AB would move the point E in the direction Eb parallel to AB, and an alteration in the bearing would move it in the direction Eb′ perpendicular to AB. Now the former change would not tend effectually to decrease the distance AE, but the latter would be almost wholly exerted in producing that effect. Again, the length of the line BC would be considerably changed without altering essentially the bearing; the former alteration would tend greatly to decrease the distance AE, but the latter would not produce so sensible an effect. Similar remarks may be made on the changes in the other bearings and distances, but it does not appear to be necessary to enter more largely on this subject.

It remains now to determine the proportion of the lines BB′ CC″, DD‴, &c. To do this we shall observe that in measuring the lengths of any lines the errors would probably be in proportion to their lengths. These supposed errors must however be decreased on those lines where the effect in correcting the error of the survey would be small, by the second and fourth principles.

In observing the bearings of all the boundary lines, equal errors are liable to be committed; however it will be proper, by the third and fourth principles, to suppose the error greater or less,

in proportion to the greater or less effect it would produce in correcting the error of the survey.

Now the error of an observed bearing being given, as for example GFI, the change of position GI of the end of the line G would be proportional to the length of the line FG (=FI), so that the supposed errors both in the length and in the bearing of any boundary line, would produce changes in the position of the end of it proportional to its length. There appears therefore a considerable degree of probability in supposing the lines BB′, C′, C″, D″ DD‴, &c. to be respectively proportional to the lengths of the boundary lines AB, BC, CD, &c. The main point to be ascertained before adopting this hypothesis is whether a due proportion of the error of the survey is thrown on the bearings and lengths of the sides. Now it is plain by this hypothesis, that the error in any boundary line is supposed to be wholly in the bearing if the line be perpendicular to AE, and wholly in its length when parallel to AE; and if the length be the same in both cases, the change of position of the end of the line would in both cases be exactly equal. Thus, if FGH be the boundary line, GI the change of position of the point B in the former case, and GH in the latter, we should in this hypothesis have GI=GH.



To show the probability of this hypothesis it may be observed, that in measuring the length of a line FGH of six or eight chains of 50 links each, an error of *one* link might easily be committed by the stretching of the chain, or the unevenness of the surface. This error would be about $\frac{1}{350}$ of the whole length. If we therefore suppose GI to be $\frac{1}{350}$ of FG, the angle GFI would be about 10′. Now, with such instruments as are generally made use of by surveyors, it is about as probable that an error of 10′ was made in the bearing, as that the above error of $\frac{1}{350}$ part was made in measuring the length. We shall therefore adopt it as a principle that the most probable way of apportioning the error of the survey AE is to take BB′, C′ C″, D″ D‴, &c. respectively proporportional to the boundary lines AB, BC, CD, &c.

Hence the following practical rule for correcting a survey.

Geometrically. Draw the boundary lines ABCDE by means of the observed bearings and lengths, and find the error of the survey AE, and let the quotient of AE divided by the sum of all the lines AB, BC, CD, DE, be represented by *r*. Through the angular points B, C, D, &c. draw the lines BB′, CC″, &c. parallel to AE, and in the same direction that A bears from E. Take

$BB'=r\times AB, CC''=r\times(AB+BC), DD'''=r\times(AB+BC+CD)$, &c. Then through the points A, B′ C″, D‴, &c. draw the corrected boundary lines ABCDA, which being determined, the area may be found by dividing the figure into triangles in the usual method.

The proportional parts BB′, CC″, &c. may be found expeditiously by means of a table of difference of latitude and departure, by finding the page where the sum of the lines AB+BC+CD+DE in the distance column corresponds to AE in the departure or difference of latitude column, then find AB, AB+BC, &c. in the distance column, and the corresponding numbers will be equal to BB′, CC″, DD‴, &c. respectively.

Arithmetically. The area of the field may also be found by means of the tables of difference of latitude and departure, by calculating for each of the observed bearings and lengths, the corresponding difference of latitude and departure; which may be corrected in the following manner. Add up the northings and southings, and find the difference of their sums, which will be the error of the survey in difference of latitude, which call by the same name as the least sum. Proceed in the same manner with the eastings and westings and find the error of the departure. These errors must be apportioned among the differences of latitude and departure by saying, as the sum of the observed distances (AB+BC+CD+DE) is to any particular distance as (AB), so is the above error in the difference of latitude or departure to the corresponding correction of the difference of latitude or departure depending on that distance*. The corrections being thus calculated and applied to the corresponding differences of latitude and departure, by adding when of the same name, and subtracting when of different names, will give the differences of latitude and departure, from which the area may be calculated by the usual rules.

This last method being made use of for calculating the area in the proposed question, the error in the difference of latitude is found 0.10 N., and the error of the departure 0.08 E., and the sum of all the distances AB+BC, &c. =161.6, as in Table 1.

$$\text{Hence}\quad \left.\begin{array}{r} 161.6:0.10::40:0.02 \\ ::25:0.02 \\ \text{\&c.}\ \ \text{\&c.} \end{array}\right\}\ \text{The corrections of difference of lat. as in Table 2.}$$

$$\text{And}\quad \left.\begin{array}{r} 161.6:0.08::40:0.02 \\ 25:0.01 \\ \text{\&c.}\ \ \text{\&c.} \end{array}\right\}\ \text{The corrections of departure as in Table 2.}$$

The corrections of Table 2 being connected with the corresponding number of Table 1 will give the corrected differences of

* These corrections may also be calculated by means of the table of difference of latitude and departure.

latitude and departure as in Table 3. By these corrected values, the areas were calculated as in Table 4. The method of doing this being taught in books of surveying, it would be useless to repeat it here.

Table 1. Traverse Table.						Table 2. Corrections.		Table 3. Correct diff. of latitude and departure.				Table 4. Calculation of the Areas.			
Course.	Dist.	N.	S.	E.	W.	N.	E	N.	S.	E.	W.	Dep.	Dep.	North Areas.	South Areas.
N. 45° E.	40	28.28		28.28		.02	.02	28.30		28.30		0.00	28.30	800.89	
S. 30° W.	25		21.65		12.50	.02	.01		21.63		12.49	12.49	12.49		270.16
S. 5° E.	36		35.86	3.14		.02	.02		35.84	3.16		9.33	21.82		782.03
W.	29.6				29.60	.02	.01	0.02			29.59	38.92	48.25	0.96	
N. 20° E.	31	29.13		10.60		.02	.02	29.15		10.62		28.30	67.22	1959.46	
	161.6	57.41	57.51	42.02	42.10	.10	.08	57.47	57.47	42.08	42.08			2761.31	1052.19
			57.41		42.02									1052.19	
		Errors.	0.10		0.08									1709.12	
													Half=	854.56	Sq. perches.

So the area is 5 acres, 1 rood, 14 $\frac{56}{100}$ perches.

N. B. In a subsequent communication Mr. Bowditch informed me that he had used this method of correcting a survey several years ago.

In justice to Mr. Bowditch it is proper to observe that his solution to Question VII. No. II. was true, and would of course have been published in No. III. (as no other true solution was sent me) had it been received in due time: but the solution published in No III. was in print before his came to hand.

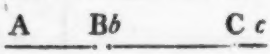
Through want of room several other learned and ingenious pieces of Mr. Bowditch's are omitted, which will enrich the future Numbers of the Analyst. ED.

ARTICLE XIV.

Research concerning the probabilities of the errors which happen in making observations, &c.

BY ROBERT ADRAIN.

The question which I propose to resolve is this: Supposing AB to be the true value of any quantity, of which the measure by observation or experiment is A*b*, the error being B*b*; what is the expression of the probability that the error B*b* happens in measuring AB?



Let AB, BC, &c. be several successive distances of which the values by measure are A*b*, *bc*, &c. the whole error being C*c*: now supposing the measures A*b*, *bc*, to be given and also the whole error C*c*, we assume as an evident principle that the most probable distances AB, BC are proportional to the measures A*b*, *bc*; and therefore the errors belonging to AB, BC are proportional to their lengths, or to their measured values A*b*, *bc*. If therefore we represent the values of AB, BC, or of their measures A*b*, *bc*, by a, b, the whole error C*c* by E, and the errors of the measures A*b*, *bc* by x, y, we must, for the greatest probability, have the equation

$$\frac{x}{a}=\frac{y}{b}.$$

Let X and Y be similar functions of a, x, and of b, y, expressing the probabilities that the errors x, y, happen in the distances a, b; and, by the fundamental principle of the doctrine of chance, the probability that both these errors happen together will be expressed by the product XY. If now we were to determine the values of x and y from the equations $x+y=\mathrm{E}$, and $\mathrm{XY}=$ maximum, we ought evidently to arrive at the equation $\frac{x}{a}=\frac{y}{b}$: and since x and y are rational functions of the simplest order possible

of a, b and E, we ought to arrive at the equation $\frac{x}{a}=\frac{y}{b}$ without the intervention of roots, in other words by simple equations; or, which amounts to the same thing in effect, if there be several forms of X and Y that will fulfil the required condition, we must choose the simplest possible, as having the greatest possible degree of probability.

Let X', Y', be the logarithms of X and Y, to any base or modulus e: and when $XY=max.$ its logarithm $X'+Y'=max.$ and therefore $\dot{X}'+\dot{Y}'=o$, which fluxional equation we may express by $X''\dot{x}+Y''\dot{y}=o$; for as X' involves only the variable quantity x, its fluxion $\dot{X}'$ will evidently involve only the fluxion of x; in like manner the fluxion of Y' may be expressed by $Y''\dot{y}$; and from the equation $X''\dot{x}+Y''\dot{y}=o$ we have $X''\dot{x}=-Y''\dot{y}$: but since $x+y=E$ we have also $\dot{x}+\dot{y}=o$, and $\dot{x}=-\dot{y}$ by which dividing the equation $X''\dot{x}=-Y''\dot{y}$, we obtain $X''=Y''$.

Now this equation ought to be equivalent to $\frac{x}{a}=\frac{y}{b}$; and this circumstance is effected in the simplest manner possible, by assuming $X''=\frac{mx}{a}$, and $Y''=\frac{my}{b}$; m being any fixed number which the question may require.

Since therefore $X''=\frac{mx}{a}$, we have $X''\dot{x}=\dot{X}'=\frac{mx\dot{x}}{a}$, and taking the fluent, we have $X'=a'+\frac{mx^2}{2a}$.

The constant quantity a' being either absolute, or some function of the distance a.

We have discovered therefore, that the logarithm of the probability that the error x happens in the distance a is expressed by $a'+\frac{mx^2}{2a}=X'$, and consequently the probability itself is

$$X=e^{X'}=e^{\left(a'+\frac{mx^2}{2a}\right)}.$$

Such is the formula by which the probabilities of different errors may be compared, when the values of the determinate quantities e, a', and m are properly adjusted. If this probability of the error x be denoted by u, the ordinate of a curve to the abscissa x, we shall have $u=e^{\left(a'+\frac{mx^2}{2a}\right)}$, which is the general equation of *the curve of probability*.

When only the maximum of probability is required, we have no need of the values of e, a' and m; it is proper however to observe that m must be negative. This is easily shown. The probability that the errors x, y, z, &c. happen in the distances a, b, c, &c. is $e^{(a'+\frac{mx^2}{2a})} \times e^{(b'+\frac{mx^2}{2a})} \times e^{(c'+\frac{mx^2}{2c})}$, &c. which is equal to $e^{(a'+b'+c'+\frac{mx^2}{2a}+\frac{my^2}{2b}+\frac{mz^2}{2c})}$, and this quantity will evidently be a maximum or minimum as its index or logarithm

$$a'+b'+c'+\frac{mx^2}{2a}+\frac{my^2}{2b}+\frac{mz^2}{2c} \text{ is a maximum or minimum,}$$

that is when $\frac{m}{2} \times \left\{\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c} \text{ \&c.}\right\} =$ a maximum or minimum.

Now when $$x+y+z+\text{\&c.} = \text{E},$$

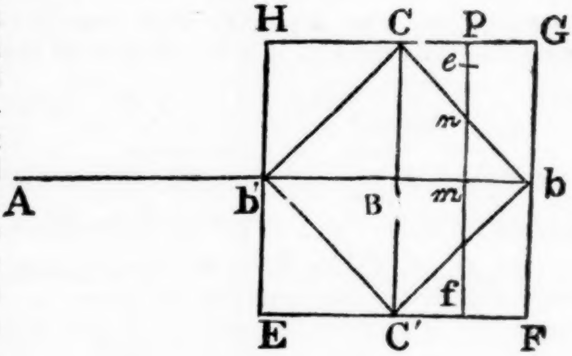
we know that $$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+ \text{\&c.} = \text{min. when} \frac{x}{a}=\frac{y}{b}=\frac{z}{c}, \text{ \&c.}$$

and therefore $$-\left\{\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+ \text{ \&c.}\right\} = \text{maximum.}$$

when $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=$ &c. it is evident therefore that m must be negative, and as we may for the case of maxima use any value of it we please we may put $m=-2$, and the probability of x in a is $u=e^{(a'-\frac{x^2}{a})}$.

If we put $\frac{m}{2a}=-1$ and $a'=f^2$, we have $u=e^{(f^2-x^2)}$ for the equation of the curve of probability; but if we suppose $f^2=o$, the ordinates u will still be proportional to their former values, and we shall have $u=e^{-x^2}$, or $u=\frac{1}{e^{x^2}}$, which is the simplest form of the equation expressing the nature of the curve of probability.

We shall now confirm what has been said by a different method of investigation.



Suppose that the length and bearing of AB are to be measured; and that the little equal straight lines B*b*, BC, re the equally probable errors, the one B*b*=B*b*′ of the length of AB, the other BC= BC′ (perpendicular to the former) of the angle at A, when measured on a circular arc to the radius AB: and let the question be to find such a curve passing through the four points *b*,C, *b*′,C′, which are equally distant from B, that, supposing the measurement to commence at A, the probability of terminating on any point of the curve may be the same as the probability of terminating on any one of the four points *b*,C, *b*′,C′.

Describe the squares *b*C *b*′C, EFGH. I say the curve sought must pass within the greater square EFGH, but without the less square *b*C *b*′C′.

Let *mne*P be drawn parallel to BC; and since the probabilities of the indefinitely little equal errors BC, *m*P, are ultimately in the ratio of equality; but the probability of the error B*m* in the distance is less than the probability of the error *o* at B, (for it is self evident that the greater the error is, the less is its probability) therefore, by the laws of chance, the probability of terminating on P is less than that of terminating on C, and therefore the point P is without the curve sought.

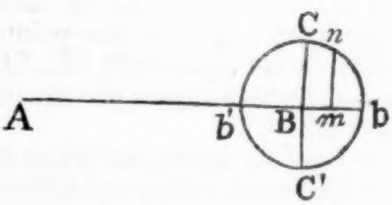
By the same argument we prove that *b*G is without the curve.

Again, since the sum of the two errors B*m*, *mn*, in distance and bearing, is together equal to the error B*b*, it follows that the probability of terminating on *n* is greater than that on *b*: for it certainly is more reasonable to suppose that each of two equal sources of error should produce a part of the whole error B*m*+*mn*=B*b*, than that the whole error B*b* should be produced from one of these sources alone, without any assistance from the other.

The same thing may also be shown thus, the probability of *mn* is the same as if it were reckoned on BC from B, and therefore the probability of *mn* is greater than that of *mb*, because any particle of error in B*b* or BC is always less probable as that particle

is farther removed from the point B, and of course the point n is within the curve; and therefore the curve must fall without the square $bC\ b'C'$. This curve therefore passes through the four points b,C, b',C' equally distant from B, and lies in the four triangles bGC, CHb', b'EC$'$, and C$'$Fb.

Farther, the curve in question ought evidently to be continuous, and have its four portions similar which lie in the four triangles bGC, CHb', &c. Its arcs proceeding from b to C or from C to b ought to be similar to each other, and to each of those proceeding from C, b', and C$'$. It must have two and only two ordinates me, mf to the same abscissa Bm; and those ordinates must be equal, the one positive, the other negative. The value of the ordinate must be the same whether the error Bm be positive or negative, that is in excess, or defect. The equation of the curve must therefore have two equal values of the ordinate $y=ne=nf$ to the same abscissa $=x$; and the abscissa x must have two equal values to the same value of the ordinate y. Lastly, the curve must be the simplest possible having all the preceding conditions, and must consequently be the circumference of a circle having its centre in B.



Now let us investigate the probability of the error Bm $=x$, and of $mn=y$.

Let X and Y be two similar functions of x and y denoting those probabilities, X$'$, Y$'$ their logarithms, then $X\times Y=$constant, or $X'+Y'$,$=$ constant, and therefore $\dot{X}'+\dot{Y}'=o$, or $X''\dot{x}+Y''\dot{y}=o$, whence $X''\dot{x}=-Y''\dot{y}$.

But $x^2+y^2=r^2=Bb^2$ therefore $x\dot{x}=-y\dot{y}$, by which dividing $X''\dot{x}=-Y''\dot{y}$, we have $\frac{X''}{x}=\frac{Y''}{y}$; and therefore, by a fundamental principle of similar functions, the similar functions $\frac{X''}{x}$ and $\frac{Y''}{y}$ must be each a constant quantity: put then $\frac{X''}{x}=n$, and we have $X''\dot{x}=nx\dot{x}$, that is $\dot{X}'=nx\dot{x}$, and the fluent is $X'=C+\frac{nx^2}{2}$; in like manner we find $Y'=C+\frac{ny^2}{2}$, and therefore the probabilities themselves are $e^{C+\frac{nx^2}{2}}$, and $e^{C+\frac{ny^2}{2}}$, in which n ought to be negative, for the probability of x grows less as x grows greater.

If now we put the constant quantities C and n equal to a', and $\frac{m}{2a}$ we have $u=e^{\left(a'+\frac{m}{2a}x^2\right)}$ as before.

As the application of this formula to maxima and minima does not require the value of a' or m we shall suppose $a'=o$, and $m=-2$, in which case we have $u=e^{\frac{-x2}{a}}$.

If there be only one quantity to which the errors relate, we may put $C=o$, and $\frac{n}{2}=-1$, in which case $u=e^{-x^2}$, in which u is the probability of the error x, or the ordinate of the curve of probability to the abscissa x.

Suppose now that the equally probable errors Bb, Bc are in any proposed ratio of 1 to p; let Bb and Bc

be expressed by x and X, and supposing that $e^{\frac{-x^2}{a}}$ is the probability of x, I say the probability of X will be $e^{\frac{-X^2}{p^2a}}$.

For since $1:p::x:X$, therefore $x=\frac{X}{p}$, $x^2=\frac{X^2}{p^2}$, and $e^{\frac{-x^2}{a}}=e^{\frac{-X^2}{p^2a}}$.

In this case the curve of *equal probability* is an ellipsis. We shall now show the use of this theory in the solution of the following problems.

PROBLEM I.

Supposing a, b, c, d, &c. to be the observed measures of any quantity x, the most probable value of x is required.

SOLUTION.

The several errors are $x-a$, $x-b$, $x-c$, $x-d$, &c. and the logarithms of their probabilities are, by what has just been shown, $-(x-a)^2$, $-(x-b)^2$, $-(x-c)^2$, &c. therefore $(x-a)^2+(x-b)^2+(x-c)^2+(x-d)^2+$ &c. $=$min.

The fluxion of this divided by $2\dot{x}$ gives us $x-a+x-b+x-c+x-d$, &c. $=o$: let n be the number of terms and we have $nx=a+b+c+d$, &c. to n terms, therefore $x=\frac{a+b+c+d, \text{ \&c.}}{n}=\frac{s}{n}$ putting s for the sum of a, b, c, &c.

Hence the following rule:

Divide the sum of all the observed values by their number, and the quotient will be the most probable value required.

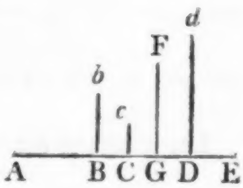
This rule coincides exactly with that commonly practised by astronomers, navigators, &c.

It is worthy of notice, that, according to the solution given above, the sum of all the errors in excess is precisely equal to the sum of all the errors in defect; in other words, the sum of all the errors is precisely equal to o, each error being taken with its proper sign: this is evident from the equation $(x-a)+(x-b)+(x-c)+(x-d)+$ &c. $=o$.

PROBLEM II.

Given the observed positions of a point in space, to find the most probable position of the point.

SOLUTION.



On any fixed plane let fall perpendiculars from all the points of position, and also from the point sought, meeting the plane in b, c, d, &c. and in F, and on the straight line AE given in position in this plane let fall the perpendiculars bB, cC, &c. Take A any fixed point in AE, and let AB, AC, AD, &c. be denoted by a, b, c, &c. Bb, Cc, Dd, &c. by a', b', c', &c. the altitudes at b, c, d, &c. by a'', b'', c'', &c. also let the sums of a, b, &c. a', b', &c. a'', b'', &c. be denoted by s, s', s'', and the number of given points by n; finally, let the three co-ordinates of the point sought, viz. AG, GF, and the altitude of the point above the plane AGF, be denoted by x, y, and z.

Now the square of the distance from F to b is $(x-a)^2+(y-a')^2$, and the difference of the altitudes at F and b is $z-a''$, therefore the square of the first error in distance is

$$(x-a)^2+(y-a')^2+(z-a'')^2$$

The square of the 2d is $(x-b)^2+(y-b')^2+(z-b'')^2$

of the 3d $(x-c)^2+(y-c')^2+(z-c'')^2$

&c. &c.

By the preceding theory, the probability of all these errors will be a maximum when the sum of their squares is a minimum, therefore, since each of the three quantities x, y, and z, is independent, the three following expressions must each be a minimum, viz.

$$(x-a)^2+(x-b)^2+(x-c)^2+\text{ \&c.} = \text{min.}$$
$$(y-a')^2+(y-b')^2+(y-c')^2+\text{ \&c.} = \text{min.}$$
$$(z-a'')^2+(z-b'')^2+(z-c'')^2+\text{ \&c.} = \text{min.}$$

These three equations put into fluxions and divided by $2\dot{x}$, $2\dot{y}$, and $2\dot{z}$, respectively, become

$$x-a+x-b+x-c+\text{ \&c.}=o,$$
$$y-a'+y-b'+y-c'+\text{ \&c.}=o,$$
$$z-a''+z-b''+z-c''+\text{ \&c}=o.$$

Whence, $x=\dfrac{a+b+c,\text{ \&c.}}{n}$, $y=\dfrac{a'+b'+c',\text{ \&c.}}{n}$, $z=\dfrac{a''+b''+c''+\text{ \&c.}}{n}$

that is, $x=\dfrac{s}{n}$, $y=\dfrac{s'}{n}$, $z=\dfrac{s''}{n}$.

Whence this rule: divide the sum of each system of ordinates by the number of given points, and the three quotients will be the three orthosonal co-ordinates of the most probable point required.

From this solution we may deduce the following remarkable consequences.

I. The point sought is so situated that the sum of the errors estimated in any direction whatever is precisely equal to o.

II. The point sought is precisely in the centre of gravity of all the given points, those points being supposed all equal: this is easily shown.

Let p be the mass at any point, and $np=M$ will be the whole mass at all the points, then we have

$$x=\frac{a+b+c\text{ \&c.}}{n}=\frac{ap+bp+cp+\text{ \&c.}}{np}=\frac{a.p+b.p+c.p\text{ \&c.}}{M}$$
$$y=\frac{a'+b'+c'\text{ \&c.}}{n}=\frac{a'p+b'p+c'p\text{ \&c.}}{np}=\frac{a'.p+b'.p+c'.p\text{ \&c.}}{M}$$
$$z=\frac{a''+b''+c''\text{ \&c.}}{n}=\frac{ap+bp+cp\text{ \&c.}}{np}=\frac{a''.p+b''.p+c''.p\text{ \&c.}}{M}$$

and these last values of x, y, z, are well known to be the three proper expressions for the three rectangular co-ordinates of the centre of gravity of all the equal quantities of matter p, p, p, &c. Hence it follows that if a point be sought such that the sum of the squares of its distances from any number of fixed points may be a minimum, the point required will be the centre of gravity of all the fixed points.

Hence also, if a point be sought such that when the squares of its distances from any number of fixed points are multiplied by the fixed numbers p, p', p'', &c. respectively, the sum may be a minimum; the required point will be precisely in the centre of gravity of all the fixed points, the quantities of matter at those points being supposed p, p', p'', respectively.

III. The data of prob. ii. being still supposed; if the *locus* of a

point be required, such, that a point situated any where on it may have an equal degree of probability to be the point sought, we may determine the *locus* from the original formulas of the preceding solution.

We must evidently have the following equation,

$$\left\{\begin{array}{l}(x-a)^2+(x-b)^2+(x-c)^2+\text{\&c.}\\(y-a')^2+(y-b')^2+(y-c')^2+\text{\&c.}\\(z-a'')^2+(z-b'')^2+(z-c'')^2+\text{\&c.}\end{array}\right\}=n\mathrm{D}^2=\text{const.}$$

that is

$$\left\{\begin{array}{l}nx^2-2sx+a^2+b^2+c^2+\text{\&c.}\\ny^2-2s'y+a'^2+b'^2+c'^2+\text{\&c.}\\nz^2-2s''z+a''^2+b''^2+c''^2+\text{\&c.}\end{array}\right\}=n\mathrm{D}^2,$$

which, by putting a^2+b^2 &c. $+a'^2+b'^2$ &c. $+a''^2-$ &c. $=n'\mathrm{D}^2$, dividing by n, and putting x', y', z' for the values of $\frac{s}{n}, \frac{s'}{n}, \frac{s''}{n}$, becomes

$$\left\{\begin{array}{l}x^2-2x'x\\y^2-2y'y\\z^2-2z'z\end{array}+\mathrm{D}'^2\right\}=\mathrm{D}^2;$$

and this last by completing the squares, transposing D'^2, and putting $\mathrm{D}^2-\mathrm{D}'^2+x'^2+y'^2+z'^2=r^2$, is converted into

$$(x'-x)^2+(y'-y)^2+(z'-y)^2=r^2,$$

which is manifestly the equation of a spherical surface having its radius r, and its centre in the point of greatest probability.

If therefore the locus of a point be required, such, that the sum of the squares of its distances from any number of fixed points may be a constant quantity, the locus sought will be a spherical surface having its centre in the centre of gravity of all the fixed points considered as equal to one another.

Hence also, if the *locus* of a point be required, such, that when the squares of its distances from any number of fixed points are respectively multiplied by the fixed numbers p, p', p'', &c. the sum of all the products may be a constant quantity, the *locus* sought will be the surface of a sphere having its centre in the common centre of gravity of all the points; the quantities of matter in the several points being respectively p, p', p'', &c.

IV. If the *locus* of equal probability (still retaining the data of prob. ii.) be restricted to a given surface, it is clear that the *locus* sought will be the line which is the common intersection of the given surface, and of a spherical surface having its centre in the point of greatest probability: and the equations of the given surface, and of the spherical surface, when referred to the same system of rectangular co-ordinates, will be the two equations of the *locus* required. If the given surface were plane, or spherical,

the *locus* of equal probability would be the circumference of a circle.

And therefore, if the *locus* of a point moving on a given surface be required, such, that when the squares of its distances from any number of fixed points are multiplied respectively by the numbers p, p', p'', &c. the sum of all the products may be a constant quantity; the *locus* sought will be the common intersection of the given surface, and of a spherical surface having its centre in the common centre of gravity of all the points, their quantities of matter being supposed to be expressed by the fixed numbers p, p', p'', &c. respectively.

V. The same data still remaining, we may also determine the points of greatest and least probability on any given line or surface.

Let $V=o$ be the equation of the given line or surface, referred to the same system of co-ordinates x, y, z. In this case let the fluxion of two equations $V=o$,

and $$\left.\begin{cases}(x-a)^2+(x-b)^2+(x-c)^2 \text{ \&c.}\\ (y-a')^2+(y-b')^2+(y-c')^2 \text{ \&c.}\\ (z-a'')^2+(z-b'')^2+(z-c'')^2 \text{ \&c.}\end{cases}\right\} = \text{max. or min.}$$

be taken; and exterminating any one of the three fluxions $\dot{x}, \dot{y}, \dot{z}$, let the co-efficients of the other two fluxions in the resulting equation be put each $=o$: this will give two equations, from which and the equation $V=o$, the values of x, y, and z may be determined by the common rules of algebra.

We may also give a geometrical plan of solution as follows; let a straight line be drawn from the point of greatest probability perpendicular to the given line or surface, and the points of intersection will be those of the maxima or minima required.

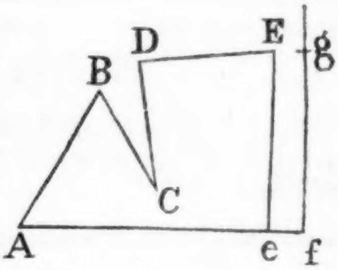
In the very same manner we determine the position of a point on a given line or surface, such, that when the squares of its distances from any number of fixed points are respectively multiplied by p, p', p'', &c. the sum of all the products may be a maximum or minimum.

From the centre of gravity of all the fixed points having the quantities of matter p, p', p'', &c. let a straight line be drawn perpendicularly to the given figure, and the intersections will give the points of maxima and minima required.

PROBLEM III.

To correct the Dead Reckoning at sea, by an observation of the latitude.

SOLUTION.



Let ABCDE be a traverse of which the difference of latitude, and departure, according to the dead reckoning are Ae, Ee; the true difference of latitude being Af; and let fg be parallel to eE.

Now the positions of the points B, C, D, &c. must be changed in such a manner, that the last point E may fall somewhere on the true parallel fg, and the probability of all these changes must be a maximum.

Let a, b, c, &c. A, B, C, &c. be the lengths and bearings of AB, BC, &c. the radius being unity; x, y, z, &c. X, Y, Z, &c. the motions or translations of the angular points B, C, D, &c. in the directions AB, BC, &c. and in directions perpendicular to the former; D', D'', D''', &c. L', L'', L''', &c. the several changes in departure and latitude, and $L = ef =$ the whole error in latitude.

The several departures are a, sin. A, b, sin. B, &c. if therefore a, and A were variable, the fluxion of the departure would evidently be

$$\sin. A.\dot{a} + a. \cos. A.\dot{A},$$

which by putting x and X for $\dot{a}$ and $a.\dot{A}$, &c. gives us the equations

$$\begin{aligned} D' &= \sin. A.x + \cos. A.X, \\ D'' &= \sin. B.y + \cos. B.Y, \\ D''' &= \sin. C.z + \cos. C.Z, \\ &\&c. \qquad \&c. \end{aligned}$$

The several differences of latitude are a. cos. A, b. cos. B, &c. and because when a, and A are variable the fluxion of a. cos. A is

$$\cos. A.\dot{a} - a. \sin. A.\dot{A},$$

therefore putting x and X for $\dot{a}$, and $a.\dot{A}$, &c. we have the following equations,

$$\begin{aligned} L' &= \cos. A.x - \sin. A.X, \\ L'' &= \cos. B.y - \sin. B.Y, \\ L''' &= \cos. C.z - \sin. C.Z, \\ &\&c. \qquad \&c. \end{aligned}$$

Now the sum of the translations of all the angular points B, C, D, &c. in the direction ef, must be equal to ef; if therefore we reckon all the bearings one way round from Af, we shall have

I. $\cos. A.x + \cos. B.y + \&c. - \sin. A.X - \sin. B.Y + \&c. = L$: and by the preceding theory, if 1 to p be the ratio of the equally probable linear errors in any proposed distance, the former in the direction of that distance, the latter at right angles to the former, we have

II. $$\frac{x^2}{a} + \frac{y^2}{b} + \&c. + \frac{X^2}{p^2a} + \frac{Y^2}{p^2a} + \&c. = \text{min}.$$

Put now the equations I, and II, into fluxions, and having multiplied the former by m, and the latter by $-\frac{1}{2}$, we shall have, by addition,

$$(m\cos. A - \frac{x}{a}) \times \dot{x} + (m\cos. B - \frac{y}{b}) \times \dot{y} + \&c. - (m\sin. A + \frac{X}{p^2a})\dot{X} - (m\sin. B + \frac{Y}{p^2b}) \times \dot{Y}, \&c. = o.$$

This equation is satisfied by making the co-efficients of the several fluxions each $= o$, whence we obtain the following equations,

$x = ma.\cos. A$, $y = mb.\cos. B$, &c. $X = -mp^2a.\sin. A$, $Y = -mp^2b.\sin. B$, &c.

These equations show us that the several motions of the angular points A,B,C, &c. in the directions of the lengths are directly as the several differences of latitude, and that their motions in directions perpendicular to the former are directly as the departures.

By substituting for x, y, X, Y, &c. their values as just determined we obtain the several corrections in latitude and departure as follows:

$L' = mp^2a\sin.^2A + ma\cos.^2A$ $\quad$ $D' = (1-p^2)ma\sin. A.\cos. A$

$L'' = mp^2b\sin.^2B + mb\cos.^2B$ $\quad$ $D'' = (1-p^2)mb\sin. B.\cos. B$

$L''' = mp^2c\sin.^2C + mc\cos.^2C$ $\quad$ $D''' = (1-p^2)mc\sin. C.\cos. C$

&c. &c. &c. &c.

These equations, by putting $p^2 = 1 + r$, become

$L' = ma + mra\sin.^2A$ $\quad$ $D' = rma\sin. A.\cos. A$

$L'' = mb + mrb\sin.^2B$ $\quad$ $D'' = rmb\sin. B.\cos. B$

&c. &c. &c. &c.

Which we may also express more simply thus,

$L' = ma + \frac{1}{2}mra$ ver. sin. $2A$ $\quad$ $D' = \frac{1}{2}rma\sin. 2A$

$L'' = mb + \frac{1}{2}mrb$ ver. sin. $2B$ $\quad$ $D'' = \frac{1}{2}rmb\sin. 2B$

&c. &c. &c. &c.

And the value of m is to be derived from the equation

$$m \left\{ a+b+ \&c. + \tfrac{1}{2}r(a \text{ ver. sin. } 2A + b \text{ ver. sin. } 2B \&c.) \right\} = L.$$

But when $p=1$ or $r=o$, which is at once the simplest and most probable value, we have

$$m = \frac{L}{a+b+c+ \&c.}$$

also $L' = ma$, $L'' = mb$, $L''' = mc$, &c.
and $D' = o$, $D'' = o$, $D''' = o$, &c.

These equations furnish the following practical rule for correcting dead reckoning. *Say, as the sum of all the distances in the traverse is to each particular distance, so is the whole error in latitude to the correction of the latitude corresponding to said distance; those corrections in latitude being always applied in such a manner as to diminish the whole error in latitude; but no corrections whatever must be applied to the several departures by account; and the differences of longitude are to be deduced from the correct differences of latitude and the corresponding departures by dead reckoning.*

But if, as is commonly practised, the whole difference of longitude is to be obtained at once by finding the whole departure, the corrections for the particular differences of latitude are unnecessary; and the rule for obtaining the correct difference of longitude is simply this:

With the departure by account, and the correct difference of latitude find the correct difference of longitude.

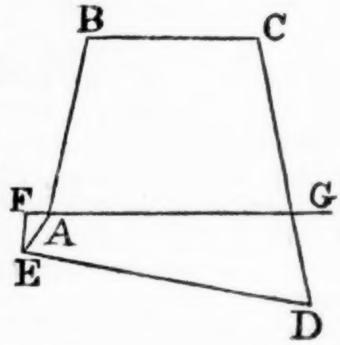
If the correct course and distance be required, they must, in like manner, be obtained from the departure by account, and the correct difference of latitude.

From this solution, it appears, that the various rules given by authors for correcting the dead reckoning are totally erroneous; and we hope they will be abandoned by all future writers on the subject.

PROBLEM IV.

To correct a survey.

SOLUTION.



Let ABCDE be a survey accurately protracted according to the measured lengths and bearings of the sides AB, BC, &c. A the place of beginning, E of ending, AG a meridian, and AF, FE the errors in latitude and departure. Now the problem requires us to make such changes in the positions of the points B, C, &c. that we may remove the errors AF, FE; in other words, that E may coincide with A, and those changes must be made in the most probable manner. We have therefore to fulfil the three following conditions.

All the changes in departure must remove the error in dep. EF.

All the changes in latitude must remove the error in lat. AF.

The probability of all these changes must be a maximum.

Let a, b, c, &c. A, B, C, &c. denote the lengths and bearings of the several sides AB, BC, &c. the radius being unity; x, y, z, &c. X, Y, Z, &c. the exceedingly small motions or translations of the angular points B, C, &c. in the directions of the sides AB, BC, &c. and in directions perpendicular to those lengths; D, L, the errors EF, FA, and D′, D″, &c. L′, L″, &c. the several corrections required in departure and latitude.

The several departures are $a.$ sin. A, $b.$ sin. B, &c. and the fluxion of $a.$ sin. A being

$$\text{sin. A.}\dot{a}+a.\text{ cos. A.}\dot{A};$$

by putting x and X for $\dot{a}$ and $a.\dot{A}$, &c. we have

$$D'=\text{sin. A.}x+\text{cos. A.X}$$
$$D''=\text{sin. B.}y+\text{cos. B.Y}$$
$$\&c. \qquad \&c.$$

And therefore

I. $\quad$ $\text{Sin. A.}x+\text{sin. B.}y+\&c.+\text{cos. A.X}+\text{cos. B.Y}+\&c.=D.$

Again, the differences of latitude are $a.$ cos. A, $b.$ cos. B, &c. and because the fluxion of $a.$ cos. A is

$$\text{Cos. A.}\dot{a}—a.\text{ sin. A.}\dot{A},$$

by putting x and X for $\dot{a}$ and $a.\dot{A}$, &c. we have

$$L'=\text{cos. A.}x—\text{sin. A.X}$$
$$L''=\text{cos. B.}y—\text{sin. B.Y}$$
$$\&c. \qquad \&c.$$

and therefore, reckoning the bearings all one way round from the meridian AG, we have

II. $\text{Cos. A}.\dot{x}+\text{cos. B}.\dot{y}+\text{\&c.}-\text{sin. A}.\dot{X}-\text{sin. B}.\dot{Y}-\text{\&c.}=L.$

Also, retaining the same signification of the letter p, we have, by the preceding theory, for the greatest probability,

III.
$$\frac{x^2}{a}+\frac{y^2}{b}+\text{\&c.}+\frac{X^2}{p^2a}+\frac{Y^2}{p^2b}+\text{\&c.}=\text{min.}$$

Now let the fluxions of the three equations I. II. III. be multiplied respectively by m, n, and $-\frac{1}{2}$, and by addition we have

$$\left.\begin{cases} m\,\text{sin.A}.\dot{x}+m\,\text{sin.B}.\dot{y}+\text{\&c.}+m\,\text{cos.A}.\dot{X}+m\,\text{cos.B}.\dot{Y}+\text{\&c.} \\ n\,\text{cos.A}.\dot{x}+n\,\text{cos.B}.\dot{y}+\text{\&c.}-n\,\text{sin.A}.\dot{X}-n\,\text{sin.B}.\dot{Y}-\text{\&c.} \\ -\frac{x}{a}.\dot{x}-\frac{y}{b}.\dot{y}-\text{\&c.}-\frac{X}{p^2a}.\dot{X}-\frac{Y}{p^2b}.\dot{Y}-\text{\&c.} \end{cases}\right\}=o.$$

This equation is satisfied by making the sum of the coefficients of each fluxion separately equal to o, whence we obtain the following equations,

$$\begin{aligned} x&=ma\,\text{sin. A}+na\,\text{cos. A}. \\ y&=mb\,\text{sin. B}+nb\,\text{cos. B}. \\ &\text{\&c.} \qquad \text{\&c.} \\ X&=mp^2a\,\text{cos. A}-np^2\,\text{sin. A}. \\ Y&=mp^2b\,\text{cos. C}-np^2\,\text{sin. B}. \\ &\text{\&c.} \qquad \text{\&c.} \end{aligned}$$

If the departures and differences of latitude of a, b, c, &c. be denoted by a', b', c', &c. a'', b'', c'', &c. we may express the several values of x, y, X, &c. thus,

$$\begin{array}{ll} x=ma'+na'' & X=p^2\times\{ma''-na'\} \\ y=mb'+nb'' & Y=p^2\times\{mb''-nb\} \\ z=mc'+nc'' & Z=p^2\times\{mc''-nc'\} \\ \text{\&c.} \quad \text{\&c.} & \text{\&c.} \quad \text{\&c.} \end{array}$$

But the proper algebraic signs of sin. A, cos. A, sin. B. cos. B, &c. must also be transferred to those values of a' b', &c. a'' b'', &c.

Putting now for x, y, X, &c. their values, we obtain the several corrections in departure and latitude as follows;

$$\begin{aligned} D'&=ma\,\text{sin.}^2\text{A}+mp^2a\,\text{cos.}^2\text{A}+(na-np^2a)\,\text{sin. A. cos. A.} \\ D''&=mb\,\text{sin.}^2\text{B}+mp^2b\,\text{cos.}^2\text{B}+(nb-np^2b)\,\text{sin. B. cos. B.} \\ D'''&= \qquad \text{\&c.} \qquad \text{\&c.} \\ L'&=na\,\text{cos.}^2\text{A}+np^2a\,\text{sin.}^2\text{A}+(ma-mp^2a)\,\text{sin. A. cos. A.} \\ L''&=nb\,\text{cos.}^2\text{B}+np^2b\,\text{sin.}^2\text{B}+(mb-mp^2b)\,\text{sin. B. cos. B.} \\ L'''&= \qquad \text{\&c.} \qquad \text{\&c.} \end{aligned}$$

Which expressions, by putting $p^2=1+r$, become

$$D' = ma + mra \cos.^2 A - nra \sin. A. \cos. A.$$
$$D'' = mb + mrb \cos.^2 B - nrb \sin. B. \cos. B.$$
&c. &c.
$$L' = na + nra \sin.^2 A - mra \sin. A. \cos. A.$$
$$L'' = nb + nrb \sin.^2 B - mrb \sin. B. \cos. B.$$
&c. &c.

and these may be transformed into the following,

$$D' = ma + \frac{ra}{2} \left\{ m + m \cos. 2A - n \sin. 2A \right\}$$
$$D'' = mb + \frac{rb}{2} \left\{ m + m \cos. 2B - n \sin. 2B \right\}$$
&c. &c.
$$L' = na + \frac{ra}{2} \left\{ n - n \cos. 2A - m \sin. 2A \right\}$$
$$L'' = nb + \frac{rb}{2} \left\{ n - n \cos. 2B - m \sin. 2B \right\}$$
&c. &c.

The values of m and n are discovered by equating the sum of the values of D', D'', &c. to D, and the sum of those of L', L'', &c. to L; m and n will therefore be found by simple equations.

In these calculations the signs of all angles in the 3d and 4th quadrants must be taken negatively, as well as the cosines of those in the 2d and 3d quadrants; according to the common rule in such cases.

Also, when the errors EF, AF are on the same side of the beginning, A, with the first departure and difference of latitude, then D and L must be taken negatively.

The simplest case of the problem is, when $p=1$, or $r=o$, which is also the most probable supposition by Prob. I; besides, this seems to agree best with the imperfections of the common instruments used in surveying.

In this case, the values of m, n, and of the required corrections in departure and latitude are as follows;

$$m = \frac{D}{a+b+c+\&c.}, \qquad n = \frac{L}{a+b+c+\&c.}$$
$$D' = ma, \quad D'' = mb, \quad D''' = mc, \ \&c.$$
$$L' = na, \quad L'' = nb, \quad L''' = nc, \ \&c.$$

Hence the following practical rule for correcting a survey:

Say, as the sum of all the distances is to each particular distance, so is the whole error in departure to the correction of the corresponding departure; each correction being so applied as to diminish the whole error in departure: proceed the same way for the corrections in latitude.

When $p=1$, as in the practical rule, the motions of the angular points B, C, D, &c. are parallel to the whole linear error EA. This appears by imagining the meridian to coincide with EA; for in this case $D=o$, and therefore

$$D'=o,\ D''=o,\ D'''=o,\ \&c.$$

which equations show that the motions of B, C, D, &c. are parallel to EA. Also because

$$L'=na,\ L''=nb,\ L'''=nc;\ \&c.$$

it follows that the motions of B, C, D, &c. in the direction EA are proportional to the several distances a, b, c, &c.

But when p is not equal to unity, the motions of B, C, &c. are not parallel to EA: when $p=o$, their motions are in the directions a, b, c, &c. because in this case the equations

$$X=p^2(ma''-na'),\ Y=p^2(mb''-nb')\ \&c.$$

become $X=o$, $Y=o$, &c.

When p is infinite, we have only to remove p^2 from the values of X, Y, &c. to those of x, y, &c. and make $p=o$; in this case the equations

$$x=p^2(ma'+na''),\ y=p^2(mb'+mb'')\ \&c.$$

become $x=o$, $y=o$, &c.

and the remaining motions $X=ma''-na'$, $Y=mb''-nb'$, &c. are manifestly perpendicular to the distances a, b, c, &c.

From this investigation it appears, that the rules hitherto given by authors for correcting a survey, are altogether erroneous, and ought to be entirely rejected. The true method here given is exemplified by Mr. Bowditch, in his solution of Mr. Patterson's question of correcting a survey; his practical rule and mine being precisely the same.

I have applied the principle of this essay to the determination of the most probable value of the earth's ellipticity, &c. but want of room will not permit me to give the investigations at this time.

ARTICLE XV.

Two excellent solutions to Mr. Garnett's prize question, page 68, have been received; the one from Mr. Garnett himself, the other from Mr. Bowditch: but the importance of the problem, as well as the improvement in the next publication of Nautical Almanacs, by Mr. Garnett, (in which the moon's right ascension will be given to minutes and seconds, instead of to minutes only as formerly) have induced the editor to postpone the decision of the prize till the next number. Such as contend for the prize must show how their solutions may be applied to the case in which the

moon's right ascension is given to the nearest second of a degree. The question is re-proposed as follows:

John Garnett, of New Brunswick, New Jersey,

offers a prize of six dollars, to be awarded by the Editor, for the best and most accurate solution to the following question:

The sun's right ascension at noon, and the moon's right ascension at noon and midnight, being always given in the Nautical Almanac for Greenwich; required, on any day, the time when the moon's centre will be on the meridian of any place whose longitude from Greenwich is known. For example, at Philadelphia, the 3d day of May, 1809. Philadelphia being supposed $5^h\ 0^m\ 55^s$ west from Greenwich.

No solution has been received to Mr. Patterson's prize question, page 69, the question is therefore re-proposed as follows:

PRIZE QUESTION.

Robert Patterson, of Philadelphia, offers a prize of six dollars, to be awarded by the Editor, at any time he may think proper, not more than five months after the publication of this number, to the author of the most simple and accurate method of finding the variation of the magnetic needle on land; without the aid of any other instrument except the common surveying-compass, and a watch that will keep time within five minutes in the week: the greatest error in variation not to exceed five minutes of a degree. The method must, of course, include directions for correcting the watch within the necessary degree of accuracy.

Mr. Bowditch sent an accurate solution to the problem of the Elastic Oval, agreeing in every particular with the results of the proposer, though the methods of solution are entirely different. Mr. Bowditch very readily consented to have the question re-proposed, hoping that some new methods of solution might be discovered. The methods both of the proposer and of Mr. Bowditch are general, and may be readily applied to more complex cases, as in the following problem of the *revolving elastic oval*, the nature of which the Geometers of America are respectfully invited to determine.

PROBLEM,

By Robert Adrain.

It is required to investigate the nature of the Elastic Oval, or the figure which a perfectly elastic circular hoop, of uniform strength and thickness, will assume, when acted on by two equal and opposite forces at the extremities of a diameter.

Another by the same.

It is required to investigate the nature of the Ellipsis Elastica Volvens, or the figure which a perfectly elastic circular hoop, of uniform strength, thickness, and density, will assume, when it revolves with uniform angular velocity about one of its diameters as an axis, in free and non-gravitating space.

PROBLEM,

By Nathaniel Bowditch.

Is the correction of the logarithm of the radius vector of the earth's orbit, arising from the disturbing force of Jupiter, rightly allowed for in Table XVI of the third edition of Lalande's Astronomy?

To assist those who have not the tables we shall observe, that the argument is expressed in thousandth parts of the whole circumference, and the corrections for each 100, beginning from 0 and ending at 1000, are as follows:

3, 4, 5, 1, —7, —11, —7, 1, 5, 4, 3.

These express the corrections of the logarithms taken to six decimal places besides the index, the mean distance of the earth from the sun being 1. The mean longitude of Jupiter for Jan. 1, 1800, being $2^s\ 21^\circ\ 8'\ 46''$, argument of Table XVI. 550. ☉'s longitude $9^s\ 9^\circ\ 54'\ 0''$. And for Jan. 1, 1801, Jupiter's longitude $3^s\ 22^\circ\ 9'\ 18''$, argument of Table XVI. 465. and ☉'s longitude $9^s\ 9^\circ\ 39'\ 41.''$

ELEGANT GEOMETRICAL PROBLEM.

Given the four sides of a trapezium, and the sum of two opposite angles, to construct the trapezium geometrically.

ARTICLE XVI.

New Questions to be answered in the next Number.

I. QUESTION 31.

By William Child, Potts Town, Pennsylvania.

In a right-angled triangle, there is given the versed sine of double the acute angle at the base equal to $\frac{3}{4}$, the radius being unity, and the area 1000, to determine the sides.

Pp. 113-114 missing?

II. QUESTION 32.

By Peter Spangler, York Town, Pennsylvania.

It is required to find a point in any given triangle, from which point if straight lines be drawn to the three angles, the triangle may be divided into three equal triangles.

III. QUESTION 33.

By James M'Ginnis, Harrisburg, Pennsylvania.

My neighbour, Mr. Fraction, says that,
$(\frac{1}{2}+\sqrt{\frac{5}{4}})+(\frac{3}{2}+\sqrt{\frac{5}{4}})=(\frac{1}{2}+\sqrt{\frac{5}{4}})\times(\frac{3}{2}+\sqrt{\frac{5}{4}})=(\frac{3}{2}+\sqrt{\frac{5}{4}})^2-(\frac{1}{2}+\sqrt{\frac{5}{4}})^2$;
it is required to prove whether his assertion be true or not.

IV. QUESTION 34.

By John Coope, Philadelphia.

Required the sides of an isosceles triangle, the equal sides containing an angle of 70°, when the area at 50 cents per square perch, and the inclosing at 60 cents per lineal perch, together, cost 500 dollars.

V. QUESTION 35.

By John Hassler, West Point, State of New York.

To make a rectangle equal to a given triangle, and having the perimeter also equal to the perimeter of the triangle.

VI. QUESTION 36.

By William Douglas, near Trenton, New Jersey.

The difference between the greater and less sides of a rhomboides is 2, the difference between their sum and the less diagonal is 5, and the difference between the squares of the diagonals is 38: the sides and diagonals are required.

VII. QUESTION 37.

By William Lenhart, Baltimore.

There were two trees 100 poles asunder, from one of which a man started to go to the other; but instead of proceeding in a direct line, he travelled in such a singular manner, that two straight lines drawn from the two trees to any point of his path, constantly formed an angle of 100 degrees: quere, the length and nature of the curve described by the man in going from the one tree to the other.